THE FOUNDER OF GROUP THEORY.

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In the very useful *Encylcopédie des Sciences Mathematiques*, tome I, volume 1 (1909), page 432, it is stated that Cauchy may be considered as the founder of the theory of substitution groups. On the contrary, Easton affirms in his *Construtcive development of group theory*, 1902, page 43, "The proper founder of group theory is Evariste Galois." Again, Pierpont has stated in the *Bulletin of the American Mathematical Society*, vol. I (1895), page 196, "in the present brief note I cannot vindicate Lagrange's right to the title of creator of the theory of substitutions; but I hope by presenting a few examples of his methods to show the importance of considering him from this point of view." A still different view is expressed by Burkhardt in his important article entitled "Die Anfaenge der Gruppen theorie und Paolo Ruffini" published in the *Abhandlungen zur Geschichte der Mathematik*, 1892, pages 119-159. In concluding this paper he says: "It will probably not be possible to determine whether the essentials of the work of his friend Abbati, which includes the first complete proof of two fundamental theorems, are due to the inspiration of Ruffini, or whether we should regard this almost forgotten man alongside with Ruffini as one of the founders of group theory."

From the preceding paragraph it is clear that within the last twenty years four different names (Cauchy, Galois, Lagrange, and Ruffini) have been given by good authorities as those of founders of the theory of groups. Among others for whom this honor has been claimed Abel is perhaps the best known. In his well known *Synopsis der hoeheren Mathematik*, 1891, page 287, Hagen says "The theory of substitution groups was founded by Abel and Cauchy, and was developed to a certain extent principally by Galois and Jordan, especially with a view to the solution of algebraic equations." On the other hand, in Maillet's, Paris, thesis, 1892, we read "The founders of the theory (of substitution groups) at least in its present form are Galois and Cauchy." These quotations may suffice to show how difficult it is to determine the founder of a large subject even of comparatively recent origin. As regards the older subjects, for instance, analytic geometry and calculus, the difficulty is generally much greater.

We are inclined to attribute the honor of starting a given big theory to an individual just as we are prone to ascribe fundamental theorems to particular men, who frequently have added only a small element to the development of the theorem. Hence the statement that a given individual founded a big theory should not generally be taken very seriously. It adds, however, a pleasant human flavor and awakens in us a noble sense of admiration and appreciation. It is also of value in giving a historical setting and brings into play a sense of the dynamic forces which have contributed to its

development instead of presenting to us a cold static scene. Observations become more inspiring when they are permeated with a sense of development.

The earliest of the names proposed as founder of the group theory is that of Lagrange, and his claim is based upon his admirable paper on the algebraic solutions of equations, published in the Berlin *Nouveaux Mémoires,* 1770 and 1771. If we accept Lagrange as the founder of group theory this subject is now about one hundred forty years old. At any rate, all the later men for whom the honor of having founded group theory is claimed were inspired by the beautiful theorems of this memoir. Among these is the fundamental theorem that the order of a subgroup is a divisor of the order of the group, which is sometimes called Lagrange's theorem.* It is scarcely necessary to add that Lagrange did not state this theorem, article 104 of his memoir, in the form in which we give it now. His language seems however practically equivalent to the given statement and is more accurate than one would naturally infer from the incorrect statement in Pierpont's review in the *Bulletin of the American Mathematical Society,* volume I (1895), top of page 198.

From the given quotations it may be inferred that Cauchy is more commonly regarded as the founder of group theory than any of the others. The opening sentence of the preface of Burnside's work on this subject, "The theory of groups of finite order may be said to date from the time of Cauchy" is in accord with such an inference. We proceed to consider some reasons for this conclusion. In the first place, it is desirable to emphasize the fact that Cauchy's contributions to group theory may be conveniently divided into two parts which are separated by a period of about thirty years. The first of these consists almost entirely of two articles published in 1815 in volume 10 of the oldest extant mathematical periodical, *Journal de l'école polytechnique,* while the second part begins with volume 3 of his *exercises d'analyse et de physique mathematique,* 1844, and closes with the numerous articles which he published in the Paris *Comptes Rendus* during 1845 and 1846.

The first of these periods is subsequent to the works of Lagrange and Ruffini but it antecedes those of Abel and Galois, while the second period is subsequent also to the works of the latter. During this second period Cauchy made his most important as well as his most extensive contributions to our subject, and these are the contributions which appear to justify the claim that he is the founder of group theory. His contributions of 1815 are scarcely as meritorious as the earlier ones by Lagrange or Ruffini, and hence these would not justify the claim that he is the founder of this theory. The assumption that Cauchy is the founder of group theory therefore implies that this theory is less than seventy years old.

One of the fundamental theorems proved by Cauchy during the

*Cf. Pincherli, *Lezioni di Algebra Complementare,* 1909, p. 44.

second period of his activity along this line is, Every group whose order is divisible by a given prime (p) must have a subgroup of order p. This is sometimes called Cauchy's theorem, and it constitutes the most difficult element in the development of Sylow's theorem. After Lagrange proved that the order of a subgroup is a divisor of the order of the group, and Ruffini established the theorem that it is not always possible to find a subgroup whose order is a given divisor of the order of the group (Ruffini's theorem), it was of great interest to establish the fact that a subgroup exists for every possible prime divisor. These three theorems, associated with the names of Lagrange, Ruffini, and Cauchy, respectively, constitute the most indispensable elements for the further development of the subject.

While Cauchy's theorem is comparable with those of Lagrange and Ruffini as regards fundamental importance for the further development of group theory, its proof demands a decidedly deeper insight into the nature and structure of a group. In this respect a fundamental theorem proved by Cauchy during his first period of group-theoretic activity is more nearly comparable with those of Lagrange and Ruffini. This theorem established the fact that the symmetric group of degree n cannot involve a subgroup whose index lies between 2 and p, where p is the largest prime which divides n. This theorem is a second extension of one due to Ruffini (1799), the first extension having been made by Abbati in 1803. It was extended still further by Bertrand, Serret, and others, and is of great importance in the study of the number of different values which a function may assume when its variables are permuted in every possible way.

Having considered the most important theorems contributed by Lagrange, Ruffini, and Cauchy towards the development of group theory it is of interest to inquire into the contributions of Galois, whose name has also been advanced as that of an individual founder of this subject. Perhaps his most important direct contribution is the introduction of the concept of modulus into group theory, a concept which Gauss had introduced into number theory about thirty years earlier (1801). In group theory, the modulus is generally known as invariant subgroup, although Jordan used the term modulus, at least indirectly, for this concept as early as 1873, in the *Bulletin of the French Mathematical Society*, volume I, page 46. In group theory this important concept is known by the following names, in chronological order: proper divisor, modulus, distinguished subgroup, invariant subgroup, monotypic subgroup, self-conjugate subgroup, normal divisor, and autojugan divisor. The multitude of names is perhaps partly due to the many different ways of approach to this concept.

Although Cauchy has contributed a number of important theorems to the development of group theory his claims as founder of this theory are more strongly supported by the fact that during the second period of his activity along this line he made the first systematic study of the theory of substitution groups, under the name of systems of conjugate substitutions.

While his developments are often prolix and involve some inaccuracies, they have placed a considerable part of the theory of substitutions into an easily accessible form and have been a source of inspiration for many of his successors. As an instance of an inaccuracy we may cite his statement (on page 443, volume 9, of the first series of his works) that a primitive group whose degree is a prime number increased by one cannot be simply transitive. It is very easily seen that the symmetric group of degree 9 can be represented as a simply transitive primitive group of degree $84=83+1$. His enumeration of the possible orders of groups of degree 6, on page 493 of the same volume, is also far from correct; but this should perhaps not surprise us in view of the large number of errors in the published enumerations of possible substitution groups.

The preceding considerations neither prove nor disprove the justice of the claim that Cauchy is the founder of group theory even if they tend to support this view. It has been our aim to exhibit a few of the elements involved in such a question, and especially to point out that many efficient workers are needed for the development of a great subject. Mathematical subjects gain in attractiveness if we can associate with them an intelligent insight into their growth and a due appreciation of the costly heritage involved in their fundamental theorems. To this end it is desirable to associate one or more founders with each of the modern subjects.

PERFECT NUMBERS.

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The theory of numbers, probably more than any other branch of mathematics, offers problems that are very easy to state and formulate completely, but extremely difficult to solve. There are many that have baffled even trained workers in this field, who have been obliged to content themselves in many cases with but partial resolutions of the questions. These very often appear as isolated, artificial problems whose solution would apparently add very little to the main body of theory. But sometimes there is an historical interest attached, which coupled with an alluring simplicity of formulation attracts investigators toward it. There is always the possibility, too, that the pursuit of solutions of even these elusive problems may lead to the discovery of mathematical relations, or processes that are new and of much more general application than to the immediate problem to be solved.

Some such justification may be necessary for research concerning the existence or relations of perfect numbers. Indeed, Fermat was led by this problem to some of his most important theorems. It is moreover a problem of much historic interest.